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A SUPERPOSITION THEOREM FOR UNBOUNDED CONTINUOUS FUNCTIONS

BY

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ABSTRACT. Let R^n be the n-dimensional Euclidean space. We prove that there are $4n$ real functions φ_q continuous on R^n with the following property: Every real function f, not necessarily bounded, continuous on R^n, can be written $f(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)) + \sum_{q=2n+2}^{4n} h(\varphi_q(x))$, $x \in R^n$, where g, h are 2 real continuous functions of one variable, depending on f.

Let $I = [0, 1]$ be the closed unit interval and let $C(I^n)$, $n = 1, 2, \ldots,$ be the Banach space of real functions continuous on the cube I^n, with the usual norm. In 1957, Kolmogorov [11] proved the following theorem, giving an elegant solution to the celebrated Hilbert's Problem 13:

For every $n = 1, 2, \ldots$ there are $n(2n + 1)$ continuous increasing functions φ_{pq} on I with the following property: Every $f \in C(I^n)$ may be written in the form

$$f(x_1 \cdots x_n) = \sum_{q=1}^{2n+1} g_q\left(\sum_{p=1}^{n} \varphi_{pq}(x_p)\right), \qquad (x_1, \ldots, x_n) \in I^n,$$

where the g_q are $2n + 1$ continuous functions of one variable, depending on f.

New research on Hilbert's Problem 13 has been carried out in three main directions:

(a) *Concerning the functions* φ_{pq}. See Fridman [4], [5], Hedberg [6], Henkin [7], Kahane [9], Kaufman [10], Sprecher [14], Vituškin [16].

(b) *Concerning the functions* g_q. See Bassalygo [1], Doss [2], [3], Kahane [8], [9], Lorentz [12], Sternfeld [15].

(c) *Concerning the basic space* I^n. See Ostrand [13]. We quote here Ostrand's theorem for we shall make use of it: Let K be a compact metric space of topological dimension n; then there are $2n + 1$ real functions φ_q, continuous on K, with the following property: Every real function f continuous on K may be written:

$$f(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in K,$$

where g is a continuous real function of one variable.

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We shall be interested here in a new situation concerning the basic space, namely, instead of I^n, we shall consider the open unit cube I_0^n, or the open unit ball $\mathfrak{B}$ in R^n, or even R^n itself, and the (possibly unbounded) continuous function on I_0^n or $\mathfrak{B}$ or R^n.

We shall prove the following:

THEOREM. *For every fixed n, there are* $2n-1$ *functions* $\psi_i, i = 1, \ldots, 2n-1$, *and* $2n+1$ *functions* φ_q, $q = 1, \ldots, 2n+1$, $4n$ *functions in all, continuous on* R^n, *taking values in the semi-open interval* $[0, 1)$, *tending to* 1 *at infinity, with the following property: Every real f, continuous on* R^n, *may be written*:

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in R^n,$$

where h, g are real functions of one variable continuous on $[0, 1)$.

The same is true if R^n is replaced throughout by I_0^n or by the open unit ball $\mathfrak{B}$ in R^n, or any space homeomorphic to these, and the various forms of the theorem are equivalent. The proof will be carried out for the open unit ball $\mathfrak{B}$.

LEMMA 1. *Let* $\mathfrak{R}$ *be the closed region bounded by concentric spheres* S, S' *in* R^n, *and let* $\delta > 0$. *Then there are* $2n+1$ *sets* $C_\delta^q = \bigcup_i C_\delta^q(i)$ *satisfying the conditions*:

(i) *For every* $q = 1, \ldots, 2n+1$, *the sets* $C_\delta^q(i)$ *form a finite family of closed disjoint sets of diameter less than* δ.

(ii) *Every* $x \in \mathfrak{R}$ *belongs to at least* $n+1$ *of the sets* C_δ^q, $q = 1, \ldots, 2n+1$.

These sets $C_\delta^q(i)$ are the intersection with $\mathfrak{R}$ of the well-known cubes considered by Kolmogorov in his classical paper [11]. There is great freedom in the choice of these sets; in the sequel of this paper we assume that they have been chosen once and for all.

LEMMA 2. *Let* $\mathfrak{R}$ *be the closed region bounded by the concentric spheres* S, S' *in* R^n, *of radius* α, α', $\alpha < \alpha'$. *There is a decreasing sequence* $\varepsilon_m \to 0$ *and there are* $2n+1$ *functions* φ_q, *continuous on* $\mathfrak{R}$, *such that*:

(1) $\alpha \leqslant \varphi_q \leqslant \alpha'$ *in the region* $\mathfrak{R}$, $q = 1, \ldots, 2n+1$,

(2) $\varphi_q(x) = \alpha$ *if and only if* $x \in S$,

(3) $\varphi_q(x) = \alpha'$ *if and only if* $x \in S'$,

(4) *for all m, and for all sets* $C_{\varepsilon_{m+1}}^q(i)$, *not meeting the* ε_m-*neighborhood of* $S \cup S'$, *we have*

$$\varphi_q\big(C_{\varepsilon_{m+1}}^q(i)\big) \cap \varphi_{q'}\big(C_{\varepsilon_{m+1}}^{q'}(i')\big) = \varnothing$$

if either $q \neq q'$ *or* $q = q'$, $i \neq i'$.

PROOF. Let δ_m be a decreasing sequence tending to 0. Choose two fixed real functions ϕ_1, ϕ_2, continuous on $\mathfrak{R}$ such that $\phi_1 = \phi_2 = \alpha$ on S, $\phi_1 = \phi_2 = \alpha'$ on S', $\alpha < \phi_1 < \phi_2 < \alpha'$ between S and S'.

Let A be the set of all $(2n + 1)$-tuples (φ_q) of functions φ_q, continuous on $\mathfrak{R}$ and satisfying the conditions

$$\phi_1 \leqslant \varphi_q \leqslant \phi_2. \tag{5}$$

Such functions φ_q satisfy necessarily conditions (1), (2), and (3) of the lemma. With the usual definition of the uniform norm $\|\cdot\|$, A is a complete metric space.

For an integer m, define the subset B_m of A as follows: The element (φ_q) of A belongs to B_m if there exists an integer $l > m$ with the property that for any sets $C_{\delta_l}^q(i)$, $C_{\delta_l}^{q'}(i')$ not meeting the δ_m-neighborhood of $S \cap S'$ we have

$$\varphi_q\big(C_{\delta_l}^q(i)\big) \cap \varphi_{q'}\big(C_{\delta_l}^{q'}(i')\big) = \varnothing$$

if either $q \neq q'$ or $q = q'$, $i \neq i'$.

We see easily that B_m is open in A.

We shall prove that B_m is dense in A. So let $(\varphi_q^0) \in A$ and let $\varepsilon > 0$ be given. We must show that we can find $(\varphi_q) \in B_m$ such that

$$\|\varphi_q - \varphi_q^0\| < \varepsilon, \qquad q = 1, \ldots, 2n + 1. \tag{6}$$

On the closed set $\mathfrak{R}_m = \mathfrak{R} \setminus \delta_m$-nbhd of $S \cup S'$ we have $\phi_1 < \phi_2$. Hence, there is $\gamma > 0$ such that

$$\phi_1(x) < \phi_2(x) - \gamma, \qquad x \in \mathfrak{R}_m.$$

Choose l so large that the variation of ϕ_1, ϕ_2 and every φ_q^0 on any set of diameter $< \delta_l$ is less than $\varepsilon/2$ and also less than $\gamma/3$. For any set $C_{\delta_l}^q(i)$ lying in $\mathfrak{R}_m$ put $\varphi_q(C_{\delta_l}^q(i)) = k^q(i)$ where the constants $k^q(i)$ are all in the open interval (α, α'), are all different, and

$$|\varphi_q(x) - \varphi_q^0(x)| < \varepsilon, \qquad x \in C_{\delta_l}^q(i) \subset \mathfrak{R}_m. \tag{6'}$$

Moreover, since $\sup_{x\in C}\phi_1(x) < \inf_{x \in C}\phi_2(x) - \gamma/3$, where C stands for $C_{\delta_l}^q(i)$, we may choose the constants $k^q(i)$ such that

$$\sup_{x\in C} \phi_1(x) < k^q(i) < \inf_{x\in C} \phi_2(x) \tag{5'}$$

for $C_{\delta_l}^q(i) \subset \mathfrak{R}_m$. Next, put $\varphi_q = \alpha$ on S, $\varphi_q = \alpha'$ on S', and then extend these φ_q, so far defined, to functions φ_q, continuous on $\mathfrak{R}$, and satisfying conditions (5) and (6). This proves that B_m is dense in A.

Now the intersection $B = \bigcap_m B_m$ is dense by Baire's theorem and, hence, is nonempty. Choose a fixed $(\varphi_q) \in B$. Then, inductively, there is a subsequence ε_m of δ_m such that for any sets $C_{\varepsilon_{m+1}}^q(i)$, $C_{\varepsilon_{m+1}}^{q'}(i')$ not meeting the ε_m-neighborhood of $S \cup S'$, we have

$$\varphi_q\big(C_{\varepsilon_{m+1}}^q(i)\big) \cap \varphi_{q'}\big(C_{\varepsilon_{m+1}}^{q'}(i')\big) = \varnothing$$

if either $q \neq q'$ or $q = q'$, $i \neq i'$.

This completes the proof of Lemma 2.

LEMMA 3. *Let $\mathfrak{R}$ be the closed ring bounded by the two concentric spheres S, S' in R^n, of radius $\alpha, \alpha', \alpha < \alpha'$. Then there are $2n+1$ functions φ_q, $q = 1, \ldots, 2n+1$, continuous on $\mathfrak{R}$, taking values in the interval $[\alpha, \alpha']$, such that $\varphi_q(x) = \alpha$ iff $x \in S$, $\varphi_q(x) = \alpha'$ iff $x \in S'$, $q = 1, \ldots, 2n+1$, with the following property:*

To every function F, continuous on $\mathfrak{R}$, vanishing on $S \cup S'$, such that $|F(x)| < M$ for $x \in \mathfrak{R}$, there corresponds a g, continuous on $[\alpha, \alpha']$, such that

$$g(\alpha) = g(\alpha') = 0, \qquad |g| \leq \frac{1}{2n+3} M,$$

and

$$\left| F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x)) \right| < \frac{2n+2}{2n+3} M \quad \textit{for } x \in \mathfrak{R}. \tag{1}$$

PROOF. We may assume $M = 1$. Since $\varepsilon_m \to 0$ (cf. Lemma 2), we may choose m so large that the oscillation of F, on any set of diameter $< \varepsilon_{m-1}$, is less than $\frac{1}{2} \cdot (2n+3)^{-1}$. Define g as follows:

If $F(x) > 0$ throughout a set $C^q_{\varepsilon_m}(i)$ not meeting the ε_{m-1}-nbhd of $S \cup S'$ put $g(\varphi_q(C^q_{\varepsilon_m}(i))) = (2n+3)^{-1}$. If $F(x) < 0$ throughout such a set, put $g(\varphi_q(C^q_{\varepsilon_m}(i))) = -(2n+3)^{-1}$. Because the closed sets $\varphi_q(C^q_{\varepsilon_m}(i))$ are disjoint, these constructions are consistent. Also, if for some $C^q_{\varepsilon_m}(i)$, the image $g(\varphi_q(C^q_{\varepsilon_m}(i))) = \pm(2n+3)^{-1}$ has been defined, then $\alpha, \alpha' \notin \varphi_q(C^q_{\varepsilon_m}(i))$, since $\varphi_q = \alpha, \alpha'$ only on $S \cup S'$, while $C^q_{\varepsilon_m}(i)$ does not meet $S \cup S'$. Therefore, it is consistent with the above construction to put $g(\alpha) = g(\alpha') = 0$. Finally, extend g to a continuous function on $[\alpha, \alpha']$, still bounded by $(2n+3)^{-1}$.

To prove that relation (1) holds we assume first that $F(x) > (2n+3)^{-1}$. By Lemma 1, there are at least $n+1$ sets $C^q_{\varepsilon_m}(i)$ containing x. By our choice of m, no such set can meet the ε_{m-1}-nbhd of $S \cup S'$ [$F = 0$ on $S \cup S'$ while $F > \frac{1}{2} \cdot (2n+3)^{-1}$ on such a set]. Therefore

$$F(x) - \sum_{q=1}^{2n+1} g(\varphi_q(x)) \leq 1 - \frac{n+1}{2n+3} + \frac{n}{2n+3} = \frac{2n+2}{2n+3} \tag{2}$$

since at least $n+1$ of the terms $g(\varphi_q(x))$ are equal to $(2n+3)^{-1}$. Also, the left side of (2) is larger than $(2n+3)^{-1} - (2n+1)/(2n+3)$, hence larger than $-(2n+2)/(2n+3)$, so that (1) is verified in this case.

The case $F(x) < -(2n+3)^{-1}$ is treated similarly.

Finally, if $|F(x)| \leq (2n+3)^{-1}$, the expression on the left side of (2) has absolute value not exceeding $(2n+2)/(2n+3)$ so that (1) holds also in this case.

The proof of Lemma 3 is now complete.

LEMMA 4. *Let* $\mathfrak{R}$ *be the closed ring bounded by the two concentric spheres* S, S', *in* R^n, *of radius* $\alpha, \alpha', \alpha < \alpha'$; *then there are* $2n + 1$ *functions* φ_q, $q = 1, \ldots, 2n + 1$, *continuous on* $\mathfrak{R}$, *taking values in the interval* $[\alpha, \alpha']$, *such that* $\varphi_q(x) = \alpha$ *for* $x \in S$, $\varphi_q(x) = \alpha'$ *for* $x \in S'$ *with the following property*:

To every function F, *continuous on* $\mathfrak{R}$ *and vanishing on* $S \cup S'$, *there corresponds a real function* g, *defined and continuous on the interval* $[\alpha, \alpha']$, *such that*

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in \mathfrak{R}.$$

Observe that for such a g *we necessarily have* $g(\alpha) = g(\alpha') = 0$

This is deduced from Lemma 3, using the very familiar Kolmogorov technique; see [11].

LEMMA 5. *Let* α_m *be an increasing sequence of positive numbers tending to* 1, *and let* S_m *be the spheres in* R^n *of center* 0, *and radius* α_m. *Then there are* $2n - 1$ *functions* ψ_i, $i = 1, \ldots, 2n - 1$, *continuous on the open unit ball* $\mathfrak{B}$ *in* R^n *of center* 0, *taking values in* $[0, 1)$, *with the following property*: *To every* f *continuous on* $\mathfrak{B}$ *there corresponds a function* h, *continuous on* $[0, 1)$, *such that*

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \qquad x \in S_{2m}, m = 1, 2, \ldots.$$

PROOF. By Ostrand's theorem [13], since $\dim S_{2m} = n - 1$, there are $2n - 1$ functions ψ_i^m, $i = 1, \ldots, 2n - 1$, continuous on S_{2m}, taking values in the interval $[\alpha_{2m}, \alpha_{2m+1}]$ with the property that every f continuous on S_{2m} may be written:

$$f(x) = \sum_{i=1}^{2n-1} h_m(\psi_i^m(x)) \quad \text{for } x \in S_{2m}$$

where h_m is continuous on $[\alpha_{2m}, \alpha_{2m+1}]$.

Let ψ_i be a continuous function on the open unit ball $\mathfrak{B}$ such that $\psi_i(x) = \psi_i^m(x)$ for $x \in S_{2m}$, and taking values in $[0, 1)$.

If now h is a continuous function on $[0, 1)$ such that $h(y) = h_m(y)$ for $y \in [\alpha_{2m}, \alpha_{2m+1}]$, then

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) \quad \text{for } x \in S_{2m}, m = 1, 2, \ldots.$$

This proves Lemma 5.

THEOREM *There are* $4n$ *functions* ψ_i, φ_q, $i = 1, \ldots, 2n - 1$, $q = 1, \ldots, 2n + 1$, *continuous on the open unit ball* $\mathfrak{B}$ *in* R^n, *taking values in the semi-open interval* $[0, 1)$, *with the following property*:

To every real function f, continuous on the open ball $\mathfrak{B}$, *there correspond two real functions h, g, continuous on* [0, 1) *such that*

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in \mathfrak{B}.$$

PROOF. By Lemma 5, starting with any increasing sequence $\alpha_0 = 0, \alpha_1, \ldots, \alpha_m, \ldots$ of real numbers tending to 1 and with the spheres S_m of center 0 and radius α_m, we have $2n-1$ functions ψ_i, $i = 1, \ldots, 2n-1$, continuous on $\mathfrak{B}$, taking values in $[0, 1)$ such that to every f continuous on $\mathfrak{B}$ there corresponds a real function h, continuous on $[0, 1)$, such that

$$(1) \qquad f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)), \qquad x \in S_{2m}, m = 0, 1, 2, \ldots.$$

Next, by Lemma 4, if $\mathfrak{R}_m$ is the closed ring bounded by the two spheres S_{2m}, S_{2m+2}, then there are $2n+1$ functions φ_q^m, $q = 1, \ldots, 2n+1$, continuous on $\mathfrak{R}_m$, taking values in the interval $[\alpha_{2m}, \alpha_{2m+2}]$ such that $\varphi_q^m(x) = \alpha_{2m}$ for $x \in S_{2m}$, $\varphi_q^m(x) = \alpha_{2m+2}$ for $x \in S_{2m+2}$ with the property that to every F continuous on $\mathfrak{R}_m$ and vanishing on $S_{2m} \cup S_{2m+2}$, there corresponds a real function g_m, defined and continuous on $[\alpha_{2m}, \alpha_{2m+2}]$, such that $g_m(\alpha_{2m}) = g_m(\alpha_{2m+2}) = 0$ and such that

$$(2) \qquad F(x) = \sum_{q=1}^{2n+1} g_m(\varphi_q^m(x)), \qquad x \in \mathfrak{R}_m.$$

Let $\varphi_q(x) = \varphi_q^m(x)$ for $x \in \mathfrak{R}_m$, $m = 0, 1, \ldots, q = 1, \ldots, 2n+1$. These functions φ_q are continuous on the open unit ball $\mathfrak{B}$ and take values in the semi-open interval $[0, 1)$.

Put now

$$(3) \qquad F(x) = f(x) - \sum_{i=1}^{2n-1} h(\psi_i(x)), \qquad x \in \mathfrak{B}.$$

Then F is continuous on $\mathfrak{B}$, and, by (1), $F(x) = 0$ for $x \in S_{2m}$, $m = 0, 1, \ldots$. Put $g(x) = g_m(x)$ for $x \in [\alpha_{2m}, \alpha_{2m+2}]$. Since $g(\alpha_{2m}) = g(\alpha_{2m+2}) = 0$, this function is unambiguously defined and is continuous on $[0, 1)$. We have, by (2),

$$F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in \mathfrak{R}_m, m = 0, 1, 2, \ldots,$$

that is $F(x) = \sum_{q=1}^{2n+1} g(\varphi_q(x))$, $x \in \mathfrak{B}$, Finally, by (3),

$$f(x) = \sum_{i=1}^{2n-1} h(\psi_i(x)) + \sum_{q=1}^{2n+1} g(\varphi_q(x)), \qquad x \in \mathfrak{B},$$

and the theorem is proved.

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